Definitions: Similarity Analysis of Counterfactuals

Philosophical Logic 2025/2026

1 Language

Definition 1.1 (Language). *Let* Var *be a non-empty set of propositional atoms* $p, q, r, \ldots$ *The language* $\mathcal{L}(\rightsquigarrow)$ *is generated by*

$$\varphi \ ::= \ p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \supset \varphi) \mid (\varphi \rightsquigarrow \varphi)$$

where $p \in \mathsf{Var}$

Definition 1.2 (Proposition expressed by a formula). *Let* M *be a model with set of worlds* W *For each* $\varphi \in \mathcal{L}(\rightsquigarrow)$*, the proposition expressed by* φ *in* M *is*

$$[\![\varphi]\!]_M \ := \ \{w \in W \ : \ M, w \vDash \varphi\}$$

When M *is fixed, we write* $[\![\varphi]\!]$

2 Similarity Frames and Models

Definition 2.1 (Similarity frame). *A* similarity frame *is a pair*

$$F \ = \ \langle W, \prec \rangle$$

such that

1. $W \neq \emptyset$ *is a set of possible worlds.*
2. *for each* $w \in W$ *there is a set* $W_w \subseteq W$ *and a binary relation* $\prec_w$ *on* W_w *satisfying*
   (a) irreflexivity*: for all* $u \in W_w$, $u \not\prec_w u$
   (b) transitivity*: for all* $u, v, z \in W_w$*, if* $u \prec_w v$ *and* $v \prec_w z$*, then* $u \prec_w z$

For $u, v \in W_w$*, the intended reading of* $u \prec_w v$ *is that* u *is more similar to* w *than* v *is.*

Definition 2.2 (Field of a world). *Let* $F = \langle W, \prec \rangle$ *be a similarity frame. For each* $w \in W$*, the* field W_w *of* w *is the domain on which* $\prec_w$ *is defined. Equivalently*

$$W_w \ = \ \{u \in W \ : \ \exists v \in W \ (u \prec_w v \text{ or } v \prec_w u)\}$$

Definition 2.3 (Reflexive closure of $\prec_w$). *Let $F = \langle W, \prec \rangle$ be a similarity frame and $w \in W$. The* reflexive closure $\preceq_w$ *of* $\prec_w$ *on* W_w *is defined by*

$$u \preceq_w v \quad \textit{iff} \quad (u \prec_w v) \textit{ or } (u = v)$$

for all $u, v \in W_w$

Definition 2.4 (Similarity model). *A* similarity model *is a triple*

$$M = \langle W, \prec, V \rangle$$

where $\langle W, \prec \rangle$ *is a similarity frame and*

$$V : \mathsf{Var} \times W \to \{0, 1\}$$

is a valuation assigning a classical truth value to each pair (p, w) *with* $p \in \mathsf{Var}$ *and* $w \in W$.

3 Truth and Logical Consequence

Definition 3.1 (Truth at a world). *Let* $M = \langle W, \prec, V \rangle$ *be a similarity model The satisfaction relation* $M, w \vDash \varphi$ *(*φ is true at w in M*) is defined inductively for* $w \in W$ *by*

$$\begin{aligned}
M, w \vDash p \quad & \textit{iff} \quad V(p, w) = 1 \\
M, w \vDash \neg\varphi \quad & \textit{iff} \quad M, w \nvDash \varphi \\
M, w \vDash \varphi \wedge \psi \quad & \textit{iff} \quad M, w \vDash \varphi \textit{ and } M, w \vDash \psi \\
M, w \vDash \varphi \vee \psi \quad & \textit{iff} \quad M, w \vDash \varphi \textit{ or } M, w \vDash \psi \\
M, w \vDash \varphi \supset \psi \quad & \textit{iff} \quad M, w \nvDash \varphi \textit{ or } M, w \vDash \psi
\end{aligned}$$

The clause for $\varphi \rightsquigarrow \psi$ *is given in definitions 4.1 and 4.4.*

Definition 3.2 (Logical consequence). *Let* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}(\rightsquigarrow)$ *We write*

$$\Gamma \vDash \varphi$$

iff for every similarity model M *and every world* w *in* M *the following holds:*

$$\textit{If } M, w \vDash \gamma \textit{ for all } \gamma \in \Gamma, \textit{ then } M, w \vDash \varphi$$

4 Counterfactual Semantics

Definition 4.1 (General similarity clause for counterfactuals). *Let* $M = \langle W, \prec, V \rangle$ *be a similarity model,* $w \in W$*, and* $\varphi, \psi \in \mathcal{L}(\rightsquigarrow)$*. Then*

$$\begin{aligned}
M, w \vDash \varphi \rightsquigarrow \psi \quad \Longleftrightarrow \quad & \forall u \in W_w \cap [\![\varphi]\!] \; \exists u' \in [\![\varphi]\!] \textit{ such that} \\
\textit{(i)} \quad & u' \preceq_w u \\
\textit{(ii)} \quad & \forall u'' \in [\![\varphi]\!] \, (u'' \preceq_w u' \Rightarrow M, u'' \vDash \psi)
\end{aligned}$$

Definition 4.2 (Limit Assumption). *Let $F = \langle W, \prec \rangle$ be a similarity frame and $w \in W$. The* Limit Assumption *holds at w iff the relation $\prec_w$ on W_w is well founded, in the sense that either of the following equivalent conditions holds*

1. *there is no infinite sequence $(u_1, u_2, u_3, \ldots)$ of worlds in W_w with $u_{n+1} \prec_w u_n$ for all $n \geq 1$*
2. *every non-empty subset $X \subseteq W_w$ has a $\prec_w$-minimal element, that is, some $u \in X$ such that there is no $v \in X$ with $v \prec_w u$*

The frame F satisfies the Limit Assumption iff it holds at every $w \in W$.

Definition 4.3 (Minimal φ-worlds). *Let $M = \langle W, \prec, V \rangle$ be a similarity model, $w \in W$, and $\varphi \in \mathcal{L}(\rightsquigarrow)$ The set of $\prec_w$-minimal φ-worlds at w is*

$$\mathrm{Min}_w(\varphi) \ := \ \{u \in W_w \cap [\![\varphi]\!] \ : \ \neg\exists v \in W_w \cap [\![\varphi]\!] \ (v \prec_w u)\}$$

Elements of $\mathrm{Min}_w(\varphi)$ *are the closest φ-worlds to w (if any)*

Definition 4.4 (Closest-worlds clause under the Limit Assumption). *Let $M = \langle W, \prec, V \rangle$ be a similarity model and $w \in W$ such that the Limit Assumption holds at w For $\varphi, \psi \in \mathcal{L}(\rightsquigarrow)$ we have*

$$M, w \vDash \varphi \rightsquigarrow \psi \quad \Longleftrightarrow \quad \forall u \in \mathrm{Min}_w(\varphi) \; M, u \vDash \psi$$

5 Frame Conditions and Corresponding Principles

We can match simple structural constraints on the similarity ordering $\prec_w$ with characteristic valid schemas for the counterfactual connective $\rightsquigarrow$.

Weak Centering (WC)	$\forall w \forall v (v \neq w \rightarrow \neg(v \prec_w w))$
(Modus Ponens for $\rightsquigarrow$)	$((\varphi \rightsquigarrow \psi) \land \varphi) \supset \psi$
Strong Centering (SC)	$\forall w \forall v (v \neq w \rightarrow w \prec_w v)$
(Conjunctive Sufficiency)	$(\varphi \land \psi) \supset (\varphi \rightsquigarrow \psi)$
Connectedness + Limit	$\forall w \forall u \forall v (u \neq v \rightarrow (u \prec_w v \lor v \prec_w u))$
	and, in addition, well-foundedness of each $\prec_w$
(CEM)	$(\varphi \rightsquigarrow \psi) \lor (\varphi \rightsquigarrow \neg\psi)$
Almost-Connectedness (AC)	$\forall w \forall u \forall v \forall z (u \prec_w z \rightarrow (u \prec_w v \lor v \prec_w z))$
(ASP)	$(\neg(\varphi \rightsquigarrow \neg\psi) \land (\varphi \rightsquigarrow \chi)) \supset ((\varphi \land \psi) \rightsquigarrow \chi)$

6 Axiomatic System P

Definition 6.1 (Axioms of **P**). *Let φ, ψ, χ range over $\mathcal{L}(\rightsquigarrow)$. The proof system* **P** *has as axioms all instances of the following schemes:*

TAUT *Every instance of a classical propositional tautology in the language $\mathcal{L}(\rightsquigarrow)$ (with $\rightsquigarrow$ treated as an additional connective).*

CI $\varphi \rightsquigarrow \varphi$.

CC $(\varphi \rightsquigarrow \psi) \wedge (\varphi \rightsquigarrow \chi) \supset (\varphi \rightsquigarrow (\psi \wedge \chi))$.

CW $(\varphi \rightsquigarrow \psi) \supset (\varphi \rightsquigarrow (\psi \vee \chi))$.

ASC $(\varphi \rightsquigarrow \psi) \wedge (\varphi \rightsquigarrow \chi) \supset ((\varphi \wedge \psi) \rightsquigarrow \chi)$.

AD $(\varphi \rightsquigarrow \chi) \wedge (\psi \rightsquigarrow \chi) \supset ((\varphi \vee \psi) \rightsquigarrow \chi)$.

Definition 6.2 (Inference rules of **P**). *The rules of inference for* **P** *are the following:*

MP (Modus Ponens) *From φ and $\varphi \supset \psi$, infer ψ.*

REA (Replacement of equivalents in antecedent)
From $\varphi \subset\supset \psi$, infer $(\varphi \rightsquigarrow \chi) \subset\supset (\psi \rightsquigarrow \chi)$

REC (Replacement of equivalents in consequent)
From $\varphi \subset\supset \psi$, infer $(\chi \rightsquigarrow \varphi) \subset\supset (\chi \rightsquigarrow \psi)$

Definition 6.3 (Derivability in **P**). *Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}(\rightsquigarrow)$. We write*

$$\Gamma \vdash_{\mathbf{P}} \varphi$$

iff there exists a finite sequence $\varphi_1, \ldots, \varphi_n$ of formulas with $\varphi_n = \varphi$ such that for each $i \leq n$ one of the following holds:

1. *φ_i is an instance of one of the axioms in definition 6.1;*
2. *$\varphi_i \in \Gamma$;*
3. *φ_i is obtained from earlier members of the sequence by one of the rules in definition 6.2.*

When $\Gamma = \emptyset$ we simply write $\vdash_{\mathbf{P}} \varphi$ and call φ a theorem *of* **P**.

Theorem 6.1 (Soundness and completeness of **P**). *For all $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}(\rightsquigarrow)$,*

$$\Gamma \vdash_{\mathbf{P}} \varphi \quad \Longleftrightarrow \quad \Gamma \models \varphi,$$

where $\models$ is the semantic consequence relation of definition 3.2 evaluated on similarity models $\langle W, \prec, V \rangle$ with the counterfactual clauses of definition 4.1 (equivalently, under the Limit Assumption, of definition 4.4).